

# Portmanteau theorem for unbounded measures

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**Abstract.** We prove an analogue of the portmanteau theorem on weak convergence of probability measures allowing measures which are unbounded on an underlying metric space but finite on the complement of any Borel neighbourhood of a fixed element.

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## 1 Introduction

Weak convergence of probability measures on a metric space has a very important role in probability theory. The well known *portmanteau theorem* due to A. D. Alexandroff (see for example Theorem 11.1.1 in Dudley [1]) provides useful conditions equivalent to weak convergence of probability measures; any of them could serve as the definition of weak convergence. Proposition 1.2.13 in the book of Meerschaert and Scheffler [3] gives an analogue of the portmanteau theorem for bounded measures on  $\mathbb{R}^d$ . Moreover, Proposition 1.2.19 in [3] gives an analogue for special unbounded measures on  $\mathbb{R}^d$ , more precisely, for extended real valued measures which are finite on the complement of any Borel neighbourhood of  $0 \in \mathbb{R}^d$ .

By giving counterexamples we show that the equivalences of (c) and (d) in Propositions 1.2.13 and 1.2.19 in [3] are not valid (see our Remarks 2.3 and 2.4). We reformulate Proposition 1.2.19 in [3] in a more detailed form adding new equivalent assertions to it (see Theorem 2.1). Moreover, we note that Theorem 2.1 generalizes the equivalence of (a) and (b) in Theorem 11.3.3 of [1] in two aspects. On the one hand, the equivalence is extended allowing not necessarily finite measures which are finite on the complement of any Borel neighbourhood of a fixed element of an underlying metric space. On the other hand, we do not assume the separability of the underlying metric space to prove the equivalence. But we mention that this latter possibility is hiddenly contained in Problem 3, p. 312 in [1]. For completeness we give a detailed proof of Theorem 2.1. Our proof goes along the lines of the proof of the original portmanteau theorem and differs from the proof of Proposition 1.2.19 in [3].

To shed some light on the sense of a portmanteau theorem for unbounded measures, let us consider the question of weak convergence of infinitely divisible probability measures  $\mu_n$ ,  $n \in \mathbb{N}$  towards an infinitely divisible probability measure  $\mu_0$  in case of the real line  $\mathbb{R}$ . Theorem VII.2.9 in Jacod and Shiriyayev [2] gives equivalent conditions for weak convergence  $\mu_n \xrightarrow{w} \mu_0$ . Among

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these conditions we have

$$\int_{\mathbb{R}} f d\eta_n \rightarrow \int_{\mathbb{R}} f d\eta_0 \quad \text{for all } f \in \mathcal{C}_2(\mathbb{R}), \quad (1.1)$$

where  $\eta_n$ ,  $n \in \mathbb{Z}_+$  are nonnegative, extended real valued measures on  $\mathbb{R}$  with  $\eta_n(\{0\}) = 0$  and  $\int_{\mathbb{R}} (x^2 \wedge 1) d\eta_n(x) < \infty$  (i.e., Lévy measures on  $\mathbb{R}$ ) corresponding to  $\mu_n$ , and  $\mathcal{C}_2(\mathbb{R})$  is the set of all real valued bounded continuous functions  $f$  on  $\mathbb{R}$  vanishing on some Borel neighbourhood of 0 and having a limit at infinity. Theorem 2.1 is about equivalent reformulations of (1.1) when it holds for all real valued bounded continuous functions on  $\mathbb{R}$  vanishing on some Borel neighbourhood of 0.

## 2 An analogue of the portmanteau theorem

Let  $\mathbb{N}$  and  $\mathbb{Z}_+$  be the set of positive and nonnegative integers, respectively. Let  $(X, d)$  be a metric space and  $x_0$  be a fixed element of  $X$ . Let  $\mathcal{B}(X)$  denote the  $\sigma$ -algebra of Borel subsets of  $X$ . A Borel neighbourhood  $U$  of  $x_0$  is an element of  $\mathcal{B}(X)$  for which there exists an open subset  $\tilde{U}$  of  $X$  such that  $x_0 \in \tilde{U} \subset U$ . Let  $\mathcal{N}_{x_0}$  denote the set of all Borel neighbourhoods of  $x_0$ , and the set of bounded measures on  $X$  is denoted by  $\mathcal{M}^b(X)$ . The expression "a measure  $\mu$  on  $X$ " means a measure  $\mu$  on the  $\sigma$ -algebra  $\mathcal{B}(X)$ .

Let  $\mathcal{C}(X)$ ,  $\mathcal{C}_{x_0}(X)$  and  $\text{BL}_{x_0}(X)$  denote the spaces of all real valued bounded continuous functions on  $X$ , the set of all elements of  $\mathcal{C}(X)$  vanishing on some Borel neighbourhood of  $x_0$ , and the set of all real valued bounded Lipschitz functions vanishing on some Borel neighbourhood of  $x_0$ , respectively.

For a measure  $\eta$  on  $X$  and for a Borel subset  $B \in \mathcal{B}(X)$ , let  $\eta|_B$  denote the restriction of  $\eta$  onto  $B$ , i.e.,  $\eta|_B(A) := \eta(B \cap A)$  for all  $A \in \mathcal{B}(X)$ .

Let  $\mu_n$ ,  $n \in \mathbb{Z}_+$  be bounded measures on  $X$ . We write  $\mu_n \xrightarrow{w} \mu$  if  $\mu_n(A) \rightarrow \mu(A)$  for all  $A \in \mathcal{B}(X)$  with  $\mu(\partial A) = 0$ . This is called *weak convergence of bounded measures* on  $X$ .

Now we formulate a portmanteau theorem for unbounded measures.

**2.1 Theorem.** *Let  $(X, d)$  be a metric space and  $x_0$  be a fixed element of  $X$ . Let  $\eta_n$ ,  $n \in \mathbb{Z}_+$ , be measures on  $X$  such that  $\eta_n(X \setminus U) < \infty$  for all  $U \in \mathcal{N}_{x_0}$  and for all  $n \in \mathbb{Z}_+$ . Then the following assertions are equivalent:*

- (i)  $\int_{X \setminus U} f d\eta_n \rightarrow \int_{X \setminus U} f d\eta_0$  for all  $f \in \mathcal{C}(X)$ ,  $U \in \mathcal{N}_{x_0}$  with  $\eta_0(\partial U) = 0$ ,
- (ii)  $\eta_n|_{X \setminus U} \xrightarrow{w} \eta_0|_{X \setminus U}$  for all  $U \in \mathcal{N}_{x_0}$  with  $\eta_0(\partial U) = 0$ ,
- (iii)  $\eta_n(X \setminus U) \rightarrow \eta_0(X \setminus U)$  for all  $U \in \mathcal{N}_{x_0}$  with  $\eta_0(\partial U) = 0$ ,
- (iv)  $\int_X f d\eta_n \rightarrow \int_X f d\eta_0$  for all  $f \in \mathcal{C}_{x_0}(X)$ ,
- (v)  $\int_X f d\eta_n \rightarrow \int_X f d\eta_0$  for all  $f \in \text{BL}_{x_0}(X)$ ,
- (vi) the following inequalities hold:
  - (a)  $\limsup_{n \rightarrow \infty} \eta_n(X \setminus U) \leq \eta_0(X \setminus U)$  for all open neighbourhoods  $U$  of  $x_0$ ,
  - (b)  $\liminf_{n \rightarrow \infty} \eta_n(X \setminus V) \geq \eta_0(X \setminus V)$  for all closed neighbourhoods  $V$  of  $x_0$ .

**Proof. (i)⇒(ii):** Let  $U$  be an element of  $\mathcal{N}_{x_0}$  with  $\eta_0(\partial U) = 0$ . Note  $\eta_n|_{X \setminus U} \in \mathcal{M}^b(X)$ ,  $n \in \mathbb{Z}_+$ . By the equivalence of (a) and (b) in Proposition 1.2.13 in [3], to prove  $\eta_n|_{X \setminus U} \xrightarrow{w} \eta_0|_{X \setminus U}$  it is enough to check  $\int_X f d\eta_n|_{X \setminus U} \rightarrow \int_X f d\eta_0|_{X \setminus U}$  for all  $f \in \mathcal{C}(X)$ . For this it suffices to show that for all real valued bounded measurable functions  $h$  on  $X$ , for all  $A \in \mathcal{B}(X)$  and for all  $n \in \mathbb{Z}_+$  we have

$$\int_X h d\eta_n|_A = \int_A h d\eta_n. \quad (2.1)$$

By Beppo-Levi's theorem, a standard measure-theoretic argument implies (2.1).

**(ii)⇒(iii):** Let  $U$  be an element of  $\mathcal{N}_{x_0}$  with  $\eta_0(\partial U) = 0$ . By (ii), we have  $\eta_n|_{X \setminus U} \xrightarrow{w} \eta_0|_{X \setminus U}$ . Since  $\eta_0|_{X \setminus U}(\partial X) = \eta_0|_{X \setminus U}(\emptyset) = 0$ , we get  $\eta_n(X \setminus U) = \eta_n|_{X \setminus U}(X) \rightarrow \eta_0|_{X \setminus U}(X) = \eta_0(X \setminus U)$ , as desired.

**(iii)⇒(ii):** Let  $U$  be an element of  $\mathcal{N}_{x_0}$  with  $\eta_0(\partial U) = 0$  and let  $B \in \mathcal{B}(X)$  be such that  $\eta_0|_{X \setminus U}(\partial B) = 0$ . We have to show  $\eta_n|_{X \setminus U}(B) \rightarrow \eta_0|_{X \setminus U}(B)$ .

Since  $B \cap (X \setminus U) = X \setminus [X \setminus (B \cap (X \setminus U))]$  and  $\eta_n|_{X \setminus U}(B) = \eta_n(B \cap (X \setminus U))$ ,  $n \in \mathbb{Z}_+$ , by (iii), it is enough to check  $\eta_0(\partial(X \setminus (B \cap (X \setminus U)))) = 0$ . First we show

$$\partial(B \cap (X \setminus U)) \subset (\partial B \cap (X \setminus U)) \cup \partial U \text{ for all subsets } B, U \text{ of } X. \quad (2.2)$$

Let  $x$  be an element of  $\partial(B \cap (X \setminus U))$  and  $(y_n)_{n \geq 1}, (z_n)_{n \geq 1}$  be two sequences such that  $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = x$  and  $y_n \in B \cap (X \setminus U)$ ,  $z_n \in X \setminus (B \cap (X \setminus U))$ ,  $n \in \mathbb{N}$ . Then for all  $n \in \mathbb{N}$  we have one or two of the following possibilities:

- $y_n \in B$ ,  $y_n \in X \setminus U$  and  $z_n \in X \setminus B$ ,
- $y_n \in B$ ,  $y_n \in X \setminus U$  and  $z_n \in U$ .

Then we get  $x \in (\partial B \cap ((X \setminus U) \cup \partial U)) \cup (\partial U \cap (B \cup \partial B)) \cup (\partial B \cap \partial U)$ . Since  $\partial B \cap ((X \setminus U) \cup \partial U) \subset (\partial B \cap (X \setminus U)) \cup \partial U$ , we have  $x \in (\partial B \cap (X \setminus U)) \cup \partial U$ , as desired.

Using (2.2) we get  $\eta_0(\partial(X \setminus (B \cap (X \setminus U)))) \leq \eta_0(\partial B \cap (X \setminus U)) + \eta_0(\partial U) = 0$ . Indeed, by the assumptions  $\eta_0(\partial B \cap (X \setminus U)) = 0$  and  $\eta_0(\partial U) = 0$ . Hence  $\eta_0(\partial(X \setminus (B \cap (X \setminus U)))) = 0$ .

**(ii)⇒(i):** Using again the equivalence of (a) and (b) in Proposition 1.2.13 in [3] and (2.1) we obtain (i).

**(iii)⇒(iv):** Let  $f$  be an element of  $\mathcal{C}_{x_0}(X)$ . Then there exists  $A \in \mathcal{N}_{x_0}$  such that  $f(x) = 0$  for all  $x \in A$  and  $\eta_0(\partial A) = 0$ . Indeed, the function  $t \mapsto \eta_0(\{x \in X : d(x, x_0) \geq t\})$  from  $(0, +\infty)$  into  $\mathbb{R}$  is monotone decreasing, hence the set  $\{t \in (0, +\infty) : \eta_0(\{x \in X : d(x, x_0) = t\}) > 0\}$  of its discontinuities is at most countable. Consequently, for all  $\tilde{U} \in \mathcal{N}_{x_0}$  there exists some  $t > 0$  such that  $U := \{x \in X : d(x, x_0) < t\} \in \mathcal{N}_{x_0}$ ,  $U \subset \tilde{U}$  and  $\eta_0(\partial U) = 0$ . (At this step we use that an element  $\tilde{U}$  of  $\mathcal{N}_{x_0}$  contains an open subset of  $X$  containing  $x_0$ .) This implies the existence of  $A$ . We show that the set  $D := \{t \in \mathbb{R} : \eta_0(\{x \in X : f(x) = t\}) > 0\}$  is at most countable. The function  $F : \mathbb{R} \rightarrow [0, \eta_0(X \setminus A)]$ , defined by

$$F(t) := \eta_0(\{x \in X \setminus A : f(x) < t\}), \quad t \in \mathbb{R},$$

is monotone increasing and left continuous. (Note that  $\eta_0(X \setminus A) < \infty$ , by the assumption on  $\eta_0$ .) Hence it has at most countably many discontinuity points, and  $t_0 \in \mathbb{R}$  is a discontinuity point of

$F$  if and only if  $F(t_0 + 0) > F(t_0)$ , i.e.,  $\eta_0(\{x \in X \setminus A : f(x) = t_0\}) > 0$ . If  $t_0 \neq 0$ , then  $\{x \in X : f(x) = t_0\} = \{x \in X \setminus A : f(x) = t_0\}$ , thus  $t_0 \neq 0$  is a discontinuity point of  $F$  if and only if  $\eta_0(\{x \in X : f(x) = t_0\}) > 0$ . Hence if  $t \in D$  then  $t = 0$  or  $t$  is a discontinuity point of  $F$ , consequently  $D$  is at most countable. Since  $f$  is bounded and  $D$  is at most countable, there exists a real number  $M > 0$  such that  $-M, M \notin D$  and  $|f(x)| < M$  for  $x \in X$ . Let  $\varepsilon > 0$ . Choose real numbers  $t_i, i = 0, \dots, k$  such that  $-M = t_0 < t_1 < \dots < t_k = M$ ,  $t_i \notin D$ ,  $i = 0, \dots, k$  and  $\max_{0 \leq i \leq k-1} (t_{i+1} - t_i) < \varepsilon$ . The countability of  $D$  implies the existence of  $t_i, i = 0, \dots, k$ . Let

$$B_i := f^{-1}([t_i, t_{i+1})) \cap (X \setminus A) = \{x \in X \setminus A : t_i \leq f(x) < t_{i+1}\}$$

for all  $i = 0, \dots, k-1$ . Then  $B_i, i = 0, \dots, k-1$ , are pairwise disjoint Borel sets and  $X \setminus A = \bigcup_{i=0}^{k-1} B_i$ . Since  $f$  is continuous, the boundary  $\partial(f^{-1}(H))$  of the set  $f^{-1}(H)$  is a subset of the set  $f^{-1}(\partial H)$  for all subsets  $H$  of  $\mathbb{R}$ . Using (2.2) this implies  $\partial(X \setminus B_i) = \partial B_i \subset f^{-1}(\{t_i\}) \cup f^{-1}(\{t_{i+1}\}) \cup \partial A$  for all  $i = 0, \dots, k-1$ . Since  $t_i \notin D, i = 0, \dots, k$ ,  $\eta_0(\partial A) = 0$  and

$$\eta_0(\partial(X \setminus B_i)) \leq \eta_0(\{x \in X : f(x) = t_i\}) + \eta_0(\{x \in X : f(x) = t_{i+1}\}) + \eta_0(\partial A),$$

we get  $\eta_0(\partial(X \setminus B_i)) = 0, i = 0, \dots, k-1$ . Since  $A \subset X \setminus B_i$ , we have  $X \setminus B_i \in \mathcal{N}_{x_0}$  for all  $i = 0, \dots, k-1$ . Hence condition (iii) implies that  $\eta_n(B_i) \rightarrow \eta_0(B_i)$  as  $n \rightarrow \infty, i = 0, \dots, k-1$ . By the triangle inequality

$$\begin{aligned} \left| \int_X f d\eta_n - \int_X f d\eta_0 \right| &= \left| \int_{X \setminus A} f d\eta_n - \int_{X \setminus A} f d\eta_0 \right| \\ &\leq 2 \max_{0 \leq i \leq k-1} (t_{i+1} - t_i) + \left| \sum_{i=0}^{k-1} t_i (\eta_n(B_i) - \eta_0(B_i)) \right|. \end{aligned}$$

Hence  $\limsup_{n \rightarrow \infty} \left| \int_X f d\eta_n - \int_X f d\eta_0 \right| \leq 2 \max_{0 \leq i \leq k-1} (t_{i+1} - t_i) < 2\varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, (iv) holds.

**(iv)  $\Rightarrow$  (v):** It is trivial, since  $\text{BL}_{x_0}(X) \subset \mathcal{C}_{x_0}(X)$ .

**(v)  $\Rightarrow$  (vi):** First let  $U$  be an open neighbourhood of  $x_0$ . Let  $\varepsilon > 0$ . We show the existence of a closed neighbourhood  $U_\varepsilon$  of  $x_0$  such that  $U_\varepsilon \subset U$  and  $\eta_0(U \setminus U_\varepsilon) < \varepsilon$ , and of a function  $f \in \text{BL}_{x_0}(X)$  such that  $f(x) = 0$  for  $x \in U_\varepsilon$ ,  $f(x) = 1$  for  $x \in X \setminus U$  and  $0 \leq f(x) \leq 1$  for  $x \in X$ .

For all  $B \in \mathcal{B}(X)$  and for all  $\lambda > 0$  we use notation  $B^\lambda := \{x \in X : d(x, B) < \lambda\}$ , where  $d(x, B) := \inf\{d(x, z) : z \in B\}$ . Since  $U$  is open, we get  $U = \bigcup_{n=1}^{\infty} F_n$ , where  $F_n := X \setminus (X \setminus U)^{1/n}$ ,  $n \in \mathbb{N}$ . Then  $F_n \subset F_{n+1}$ ,  $n \in \mathbb{N}$ ,  $F_n$  is a closed subset of  $X$  for all  $n \in \mathbb{N}$  and  $\bigcap_{n=1}^{\infty} (X \setminus F_n) = X \setminus U$ . We also have  $\eta_0(X \setminus F_n) < \infty$  for some sufficiently large  $N \in \mathbb{N}$  and  $X \setminus F_n \supset X \setminus F_{n+1}$  for all  $n \in \mathbb{N}$ , and hence the continuity of the measure  $\eta_0$  implies that  $\lim_{n \rightarrow \infty} \eta_0(X \setminus F_n) = \eta_0(X \setminus U)$ . Since  $\eta_0(X \setminus U) < \infty$ , there exists some  $n_0 \in \mathbb{N}$  such that  $\eta_0(X \setminus F_{n_0}) - \eta_0(X \setminus U) < \varepsilon$ . Set  $U_\varepsilon := F_{n_0}$ . Since  $\eta_0(X \setminus F_{n_0}) - \eta_0(X \setminus U) = \eta_0((X \setminus F_{n_0}) \setminus (X \setminus U)) = \eta_0(U \setminus F_{n_0})$ , the set  $U_\varepsilon$  is a closed neighborhood of  $x_0$ ,  $U_\varepsilon \subset U$  and  $\eta_0(U \setminus U_\varepsilon) < \varepsilon$ .

We show that the function  $f : X \rightarrow \mathbb{R}$ , defined by  $f(x) := \min(1, n_0 d(x, U_\varepsilon))$ ,  $x \in X$ , is an element of  $\text{BL}_{x_0}(X)$ ,  $f(x) = 0$  for  $x \in U_\varepsilon$ ,  $f(x) = 1$  for  $x \in X \setminus U$  and  $0 \leq f(x) \leq 1$  for  $x \in X$ .

Note that if  $x \in U_\varepsilon$  then  $d(x, U_\varepsilon) = 0$ , hence  $f(x) = 0$ . And if  $x \in X \setminus U$  then  $d(x, U_\varepsilon) \geq d(X \setminus U, U_\varepsilon) \geq 1/n_0$ , hence  $f(x) = 1$ . The fact that  $0 \leq f(x) \leq 1$ ,  $x \in X$  is obvious. To prove that  $f$  is Lipschitz, we check that

$$|f(x) - f(y)| \leq n_0 d(x, y) \quad \text{for all } x, y \in X.$$

If  $x, y \in X$  with  $d(x, y) \geq 1/n_0$  then  $|f(x) - f(y)| \leq 1 \leq n_0 d(x, y)$ . If  $x, y \in X$  with  $d(x, y) < 1/n_0$  then we have to consider the following four cases apart from changing the role of  $x$  and  $y$ :  $x \in X \setminus U$ ,  $y \in U \setminus U_\varepsilon$ ;  $x \in U_\varepsilon$ ,  $y \in U \setminus U_\varepsilon$ ;  $x, y \in U \setminus U_\varepsilon$  and the case  $x, y \in U_\varepsilon$  or  $x, y \in X \setminus U$ .

Let us consider the case when  $x, y \in U \setminus U_\varepsilon$  and  $f(x) = 1$ ,  $f(y) = n_0 d(y, U_\varepsilon)$ . Then  $d(x, U_\varepsilon) \geq 1/n_0$ ,  $d(y, U_\varepsilon) \leq 1/n_0$  and we get  $|f(x) - f(y)| = 1 - n_0 d(y, U_\varepsilon) \leq n_0 d(x, y)$ . Indeed,  $1/n_0 \leq d(x, U_\varepsilon) \leq d(x, y) + d(y, U_\varepsilon)$ . The case  $x, y \in U \setminus U_\varepsilon$  and  $f(y) = 1$ ,  $f(x) = n_0 d(x, U_\varepsilon)$  can be handled similarly. If  $x, y \in U \setminus U_\varepsilon$  and  $f(x) = n_0 d(x, U_\varepsilon)$ ,  $f(y) = n_0 d(y, U_\varepsilon)$  then

$$|f(x) - f(y)| = n_0 |d(x, U_\varepsilon) - d(y, U_\varepsilon)| \leq n_0 d(x, y).$$

Indeed, since  $U_\varepsilon$  is closed, we have  $|d(x, U_\varepsilon) - d(y, U_\varepsilon)| \leq d(x, y)$ . If  $x, y \in U \setminus U_\varepsilon$  and  $f(x) = f(y) = 1$  then  $|f(x) - f(y)| = 0 \leq n_0 d(x, y)$ .

The other cases can be handled similarly. Hence  $f \in \text{BL}_{x_0}(X)$  and we get

$$\int_X f d\eta_0 = \int_{X \setminus U_\varepsilon} f d\eta_0 \leq \eta_0(X \setminus U_\varepsilon) = \eta_0(X \setminus U) + \eta_0(U \setminus U_\varepsilon) < \eta_0(X \setminus U) + \varepsilon,$$

and  $\int_X f d\eta_n \geq \int_{X \setminus U} f d\eta_n = \eta_n(X \setminus U)$ . Hence by condition (v) we have

$$\limsup_{n \rightarrow \infty} \eta_n(X \setminus U) \leq \limsup_{n \rightarrow \infty} \int_X f d\eta_n = \int_X f d\eta_0 < \eta_0(X \setminus U) + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we get (a).

Now let  $V$  be a closed neighbourhood of  $x_0$ . Let  $\varepsilon > 0$ . As in case of an open neighbourhood of  $x_0$ , one can show that there exist an open neighbourhood  $V_\varepsilon$  of  $x_0$  such that  $V \subset V_\varepsilon$  and  $\eta_0(V_\varepsilon \setminus V) < \varepsilon$  and a function  $f \in \text{BL}_{x_0}(X)$  such that  $f(x) = 0$  for  $x \in V$ ,  $f(x) = 1$  for  $x \in X \setminus V_\varepsilon$  and  $0 \leq f(x) \leq 1$  for  $x \in X$ . Then we get

$$\begin{aligned} \int_X f d\eta_0 &= \int_{X \setminus V} f d\eta_0 = \eta_0(X \setminus V_\varepsilon) + \int_{V_\varepsilon \setminus V} f d\eta_0 \\ &\geq \eta_0(X \setminus V) - \eta_0(V_\varepsilon \setminus V) > \eta_0(X \setminus V) - \varepsilon, \end{aligned}$$

and  $\int_X f d\eta_n = \int_{X \setminus V} f d\eta_n \leq \eta_n(X \setminus V)$ . Hence by condition (v) we have

$$\liminf_{n \rightarrow \infty} \eta_n(X \setminus V) \geq \liminf_{n \rightarrow \infty} \int_X f d\eta_n = \int_X f d\eta_0 > \eta_0(X \setminus V) - \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we obtain (b).

(vi)  $\Rightarrow$  (iii): The proof can be carried out similarly to the proof of the corresponding part of Theorem 11.1.1 in Dudley [1].  $\square$

**2.2 Remark.** Assertion (v) in Theorem 2.1 can be replaced by

$$\int_X f d\eta_n \rightarrow \int_X f d\eta_0 \quad \text{for all } f \in \mathcal{C}_{x_0}^u(X),$$

where  $\mathcal{C}_{x_0}^u(X)$  denotes the set of all uniformly continuous functions in  $\mathcal{C}_{x_0}(X)$ .

**2.3 Remark.** By giving a counterexample we show that (a) and (b) in condition (vi) of Theorem 2.1 are not equivalent. For all  $n \in \mathbb{N}$  let  $\eta_n$  be the Dirac measure  $\delta_2$  on  $\mathbb{R}$  concentrated on 2 and let  $\eta_0$  be the Dirac measure  $\delta_0$  on  $\mathbb{R}$  concentrated on 0. Then  $\eta_0(\mathbb{R} \setminus V) = 0$  for all closed neighbourhoods  $V$  of 0, hence (b) in condition (vi) of Theorem 2.1 holds. But (a) in condition (vi) of Theorem 2.1 is not satisfied. Indeed,  $U := (-1, 1)$  is an open neighbourhood of 0,  $\eta_0(\mathbb{R} \setminus U) = 0$ , but

$$\eta_n(\mathbb{R} \setminus U) = \eta_n((-\infty, -1], [1, +\infty)) = 1, \quad n \in \mathbb{N},$$

hence  $\limsup_{n \rightarrow \infty} \eta_n(\mathbb{R} \setminus U) = 1$ . This counterexample also implies that the equivalence of (c) and (d) in Proposition 1.2.19 in [3] is not valid.

**2.4 Remark.** By giving a counterexample we show that the equivalence of (c) and (d) in Proposition 1.2.13 in [3] is not valid. For all  $n \in \mathbb{N}$  let  $\mu_n$  be the measure  $2\delta_{1/n}$  on  $\mathbb{R}$  and  $\mu$  be the Dirac measure  $\delta_0$  on  $\mathbb{R}$ . We have  $\mu(A) \leq \liminf_{n \rightarrow \infty} \mu_n(A)$  for all open subsets  $A$  of  $\mathbb{R}$  but there exists some closed subset  $F$  of  $\mathbb{R}$  such that  $\limsup_{n \rightarrow \infty} \mu_n(F) > \mu(F)$ . If  $A$  is an open subset of  $\mathbb{R}$  such that  $0 \in A$  then  $\mu(A) = 1$  and  $\mu_n(A) = 2$  for all sufficiently large  $n$ , which implies  $\mu(A) \leq \liminf_{n \rightarrow \infty} \mu_n(A)$ . If  $A$  is an open subset of  $\mathbb{R}$  such that  $0 \notin A$  then  $\mu(A) = 0$ , hence  $\mu(A) \leq \liminf_{n \rightarrow \infty} \mu_n(A)$  is valid. Let  $F$  be the closed interval  $[-1, 1]$ . Then  $\mu(F) = 1$  and  $\mu_n(F) = 2$ ,  $n \in \mathbb{N}$ , which yields  $\limsup_{n \rightarrow \infty} \mu_n(F) = 2$ . Hence  $\limsup_{n \rightarrow \infty} \mu_n(F) > \mu(F)$ .

## References

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